Generation of N-Photon Entangled States in a Two-Mode Jaynes–Cummings Model

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1. Entangled states

We consider the interaction of a two-level atom with two degenerate modes of a lossless cavity. We solve the corresponding Jaynes–Cummings (JC) model by an SU(2) transformation and discuss the generation of entangled states with a total number N of photons in the two modes:

$$|\Psi_N\rangle = \sum_{k=0}^{N} c_k^{(N)} |N-k,k\rangle, \qquad (1)$$

which comprise also the maximally entangled Bell-states

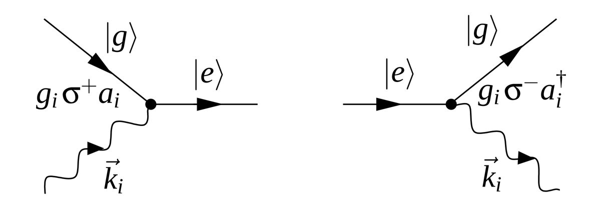
$$|\Psi_N^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle \pm |0,N\rangle\right). \qquad (2)$$

The states (1) and (2) are of great interest in the field of quantum optical lithography where they help to overcome the classical Rayleigh diffraction limit by a factor of $1/N$, N being the total number of photons in the two-mode field states.

2. Reduction to a one-mode interaction

The JC Hamiltonian for resonant interaction of a two-level atom ($|e\rangle, |g\rangle$) with two field modes (a_1, a_2), is given by $H = H_0 + H_{\mathrm{int}}$, where

$$\begin{aligned} H_0 &= \hbar\omega\left(\frac{\sigma_z + \mathbf{1}}{2} + (a_1^\dagger a_1 + a_2^\dagger a_2)\mathbf{1}\right), \\ H_{\mathrm{int}} &= \hbar\left(\sigma^+ (g_1 a_1 + g_2 a_2) + \sigma^- \left(g_1^* a_1^\dagger + g_2^* a_2^\dagger\right)\right). \end{aligned}$$



Feynman representations of the interaction operator H_{int}.

The operators for the atom are defined by $\sigma_z := |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma^+ := |e\rangle\langle g|$ and $\sigma^- := |g\rangle\langle e|$. We introduce quasi-mode operators:

$$A_1 = \gamma_1 a_1 + \gamma_2 a_2, \quad A_2 = -\gamma_2^* a_1 + \gamma_1^* a_2, \qquad (3)$$

where $\gamma_i := g_i/g$, $g := \sqrt{|g_1|^2 + |g_2|^2}$. This is an SU(2) transformation of the mode operators a_1, a_2. The transformed Hamiltonian then reads

$$\begin{aligned} H_0 &= \hbar\omega\left(\frac{\sigma_z + \mathbf{1}}{2} + (A_1^\dagger A_1 + A_2^\dagger A_2)\mathbf{1}\right), \\ H_{\mathrm{int}} &= \hbar g\left(\sigma^+ A_1 + \sigma^- A_1^\dagger\right), \end{aligned}$$

representing a JC Hamiltonian for quasi-mode one decoupled from a non-interacting quasi-mode two. The time-evolution operator in the interaction picture is then the same as for a one quasi-mode JC model:

$$U = \exp(-\mathrm{i}H_{\mathrm{int}}t/\hbar) =$$

$$\cos\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)|e\rangle\langle e| + \frac{\sin\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)}{\mathrm{i}\sqrt{A_1^\dagger A_1 + 1}} A_1 |e\rangle\langle g|$$

$$+ A_1^\dagger \frac{\sin\left(\tau\sqrt{A_1^\dagger A_1 + 1}\right)}{\mathrm{i}\sqrt{A_1^\dagger A_1 + 1}} |g\rangle\langle e| + \cos\left(\tau\sqrt{A_1^\dagger A_1}\right)|g\rangle\langle g|.$$

2.1 Properties of the quasi-mode operators

The quasi-mode operators $A_i, A_i^\dagger$, $i = 1,2$, obey the same algebra as the mode operators $a_i, a_i^\dagger$, so that two-quasi-mode Fock states (denoted by a double-ket) can be defined by

$$|n_1, n_2\rangle\!\rangle := \frac{A_1^{\dagger n_1} A_2^{\dagger n_2}}{\sqrt{n_1!\, n_2!}} |0,0\rangle\!\rangle.$$

To find the transformation between the two-mode Fock states $|n_1, n_2\rangle$ and the two-quasi-mode Fock states $|n_1, n_2\rangle\!\rangle$, we use Schwinger's oscillator model and introduce angular momentum states $|j,m\rangle$ and $|j,m\rangle\!\rangle$, where $j = (n_1 + n_2)/2$ and $m = (n_1 - n_2)/2$. We obtain the important relation between the quasi-mode and the mode Fock bases.

$$\begin{aligned} |j,m\rangle\!\rangle &= \sum_{m'=-j}^{j} D_{m'm}^{(j)}(\varphi,\vartheta,\chi)\, |j,m'\rangle, & (5) \\ |j,m\rangle &= \sum_{m'=-j}^{j} D_{m'm}^{(j)\dagger}(\varphi,\vartheta,\chi)\, |j,m'\rangle\!\rangle. & (6) \end{aligned}$$

Here $D_{m'm}^{(j)}(\varphi,\vartheta,\xi) = \exp[-\mathrm{i}(m'\varphi + m\chi)]\, d_{m'm}^{(j)}(\vartheta)$ are the Wigner D-matrix elements of the SU(2) group with arguments determined by $\varphi = \varphi_1 - \varphi_2$, $\chi = \varphi_1 + \varphi_2$, $\cos(\vartheta/2) := |\gamma_1|$, $\sin(\vartheta/2) := |\gamma_2|$, and $\gamma_i = |\gamma_i| \exp(\mathrm{i}\varphi_i)$. The action of U_{ab} on the field states is easily calculated in the quasi-mode Fock basis

$$\begin{aligned} U_{ee}(\tau)\,|j,m\rangle\!\rangle &= \cos\left(\tau\sqrt{j+m+1}\right)|j,m\rangle\!\rangle, \\ U_{ge}(\tau)\,|j,m\rangle\!\rangle &= -\mathrm{i}\sin\left(\tau\sqrt{j+m+1}\right)\left|j+\tfrac{1}{2}, m+\tfrac{1}{2}\right\rangle\!\rangle, \\ U_{eg}(\tau)\,|j,m\rangle\!\rangle &= -\mathrm{i}\sin\left(\tau\sqrt{j+m}\right)\left|j-\tfrac{1}{2}, m-\tfrac{1}{2}\right\rangle\!\rangle, \\ U_{gg}(\tau)\,|j,m\rangle\!\rangle &= \cos\left(\tau\sqrt{j+m}\right)|j,m\rangle\!\rangle, & (7) \end{aligned}$$

showing that U_{ee} and U_{gg} do not change the number of quasi-photons, whereas U_{ge} (U_{eg}) act as creation (annihilation) operators of quasi-mode one. We find for the action on the usual Fock states

$$\begin{aligned} U_{ee}(\tau)\,|j,m\rangle &= \sum_{m'=-j}^{j} C_{m'm}^{j}(\tau)\,|j,m'\rangle, \\ U_{ge}(\tau)\,|j,m\rangle &= \sum_{m'=-j-\frac{1}{2}}^{j+\frac{1}{2}} S_{m'm}^{j}(\tau)\left|j+\tfrac{1}{2}, m'\right\rangle, \\ U_{eg}(\tau)\,|j,m\rangle &= \sum_{m'=-j+\frac{1}{2}}^{j-\frac{1}{2}} \overline{S}_{m'm}^{j}(\tau)\left|j-\tfrac{1}{2}, m'\right\rangle, \\ U_{gg}(\tau)\,|j,m\rangle &= \sum_{m'=-j}^{j} \overline{C}_{m'm}^{j}(\tau)\,|j,m'\rangle, & (8) \end{aligned}$$

where we have introduced the following coefficients

$$\begin{aligned} C_{m'm}^{j}(\tau) &= \sum_{\nu=-j}^{j} \cos\left(\tau\sqrt{j+\nu+1}\right) D_{m'\nu}^{(j)} D_{\nu m}^{(j)\dagger}, \\ S_{m'm}^{j}(\tau) &= -\mathrm{i}\sum_{\nu=-j}^{j} \sin\left(\tau\sqrt{j+\nu+1}\right) D_{m',\nu+\frac{1}{2}}^{(j+\frac{1}{2})} D_{\nu m}^{(j)\dagger}, \\ \overline{S}_{m'm}^{j}(\tau) &= -\mathrm{i}\sum_{\nu=-j}^{j} \sin\left(\tau\sqrt{j+\nu}\right) D_{m',\nu-\frac{1}{2}}^{(j-\frac{1}{2})} D_{\nu m}^{(j)\dagger}, \\ \overline{C}_{m'm}^{j}(\tau) &= \sum_{\nu=-j}^{j} \cos\left(\tau\sqrt{j+\nu}\right) D_{m'\nu}^{(j)} D_{\nu m}^{(j)\dagger}. & (9) \end{aligned}$$

3. Generation of $|\Psi_N\rangle$

Let the atom-field system initially be in the state $|e;\xi\rangle$ or $|g;\xi\rangle$ and consider the states (1). We denote their pure state density matrix by $\rho_{\Psi_N} = |\Psi_N\rangle\langle\Psi_N|$.
From the time-evolved initial states $U|e;\xi\rangle$, $U|g;\xi\rangle$ we obtain the reduced field-state density matrix $\rho_{\mathrm{F}}(t)$ by tracing out the atomic degrees of freedom. The expectation value for the generation of $|\Psi_N\rangle$ then reads $\langle\rho_{\Psi_N}\rangle = \mathrm{Tr}(\rho_{\mathrm{F}}(t)\rho_{\Psi_N})$.
It follows that in order to obtain non-vanishing probabilities at time τ, the initial field state must contain at least one of the Fock states from the set

$$\begin{aligned} &\{|0,N\rangle, |1,N-1\rangle, \ldots, |N,0\rangle\} \\ &\quad \cup \{|0,N-1\rangle, |1,N-2\rangle, \ldots, |N-1,0\rangle\} \end{aligned}$$

it the atom is initially in the excited state, or from the set

$$\begin{aligned} &\{|0,N\rangle, |1,N-1\rangle, \ldots, |N,0\rangle\} \\ &\quad \cup \{|0,N+1\rangle, |1,N\rangle, \ldots, |N+1,0\rangle\} \end{aligned}$$

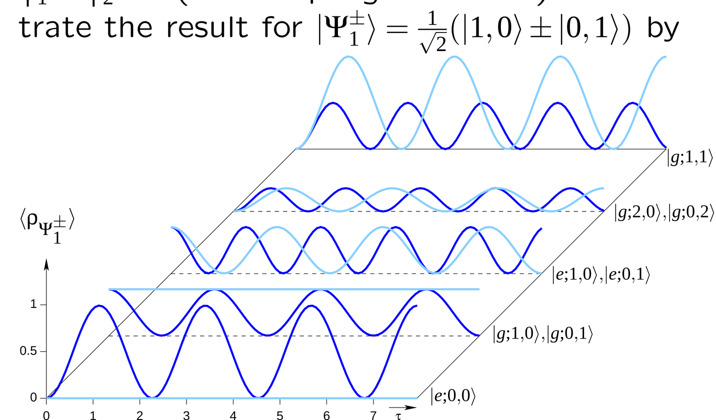
if it is in the ground state. As an example, we consider the creation of the Bell states in Eq. (2). Starting with the initial separable N-photon state $|e;N,0\rangle = \left|e;\frac{N}{2},\frac{N}{2}\right\rangle_{\mathrm{S}}$, we obtain the entangled N-photon state $\left|\Psi_N^{\pm}\right\rangle$ with probability

$$\left\langle \rho_{\Psi_N^{\pm}} \right\rangle = \frac{1}{2}\left| C_{\frac{N}{2}\frac{N}{2}}^{\frac{N}{2}}(\tau) \pm C_{-\frac{N}{2}\frac{N}{2}}^{\frac{N}{2}}(\tau) \right|^2, \qquad (10)$$

as well as the entangled $(N+1)$-photon state $\left|\Psi_{N+1}^{\pm}\right\rangle$ with probability

$$\left\langle \rho_{\Psi_{N+1}^{\pm}} \right\rangle = \frac{1}{2}\left| S_{\frac{N+1}{2}\frac{N}{2}}^{\frac{N}{2}}(\tau) \pm S_{-\frac{N+1}{2}\frac{N}{2}}^{\frac{N}{2}}(\tau) \right|^2. \qquad (11)$$

In the case of Eq. (11) the Bell states $|\Psi_{N+1}^{\pm}\rangle$ have no overlap with the initial field state $|N,0\rangle$. The probability at time τ, however, may come close to one for some particular values of the coupling constants and interaction time. In this case we may say that $|\Psi_{N+1}^{\pm}\rangle$ has been generated in a *single step* or single shot. For the case of $N = 1$, $g_1 = g_2$, $\varphi_1 = \varphi_2 = 0$ (real coupling constants) we illustrate the result for $|\Psi_1^{\pm}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle \pm |0,1\rangle)$ by



The figure shows the probability $\langle\rho_{\Psi_1^{\pm}}\rangle$ for different initial atom-field states contributing to $N = 1$. The dark-blue curves belong to the probability $\langle\rho_{\Psi_1^{+}}\rangle$ and the light-blue ones to $\langle\rho_{\Psi_1^{-}}\rangle$. On the x-axis we use the dimensionless time $\tau = gt$.

In an experimental realization the interaction time of the atom with the cavity-modes have to be adjusted to obtain a maximum. For general results and other schemes see:

References

[1] C. Wildfeuer and D. H. Schiller, quant-ph/0210138